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Integrating learning theories and application-based modules in teaching linear algebra ☆

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ABSTRACT

The research team of The Linear Algebra Project developed and implemented a curriculum and a pedagogy for parallel courses in (a) linear algebra and (b) learning theory as applied to the study of mathematics with an emphasis on linear algebra. The purpose of the ongoing research, partially funded by the National Science Foundation, is to investigate how the parallel study of learning theories and advanced mathematics influences the development of thinking of individuals in both domains. The researchers found that the particular synergy afforded by the parallel study of math and learning theory promoted, in some students, a rich understanding of both domains and that had a mutually reinforcing effect. Furthermore, there is evidence that the deeper insights will contribute to more effective instruction by those who become high school math teachers and, consequently, better learning by their students. The courses developed were appropriate for mathematics majors, pre-service secondary mathematics teachers, and practicing mathematics teachers. The learning seminar focused most heavily on constructivist theories, although it also examined

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Thematicized schema
Triad – intra
Inter
Trans
Genetic decomposition
Vector addition
Matrix
Matrix multiplication
Matrix representation
Basis
Column space
Row space
Null space
Eigenspace
Transformation

socio-cultural and historical perspectives. A particular theory, Action–Process–Object–Schema (APOS) [10], was emphasized and examined through the lens of studying linear algebra. APOS has been used in a variety of studies focusing on student understanding of undergraduate mathematics. The linear algebra courses include the standard set of undergraduate topics. This paper reports the results of the learning theory seminar and its effects on students who were simultaneously enrolled in linear algebra and students who had previously completed linear algebra and outlines how prior research has influenced the future direction of the project.

1. Research rationale

The research team of the Linear Algebra Project (LAP) developed and implemented a curriculum and a pedagogy for parallel courses in linear algebra and learning theory as applied to the study of mathematics with an emphasis on linear algebra. The purpose of the research, which was partially funded by the National Science Foundation (DUE CCLI 0442574), was to investigate how the parallel study of learning theories and advanced mathematics influences the development of thinking of high school mathematics teachers, in both domains. The researchers found that the particular synergy afforded by the parallel study of math and learning theory promoted, in some teachers, a richer understanding of both domains that had a mutually reinforcing effect and affected their thinking about their identities and practices as teachers.

It has been observed that linear algebra courses often are viewed by students as a collection of definitions and procedures to be learned by rote. Scanning the table of contents of many commonly used undergraduate textbooks will provide a common list of terms such as listed here (based on linear algebra texts by Strang [1] and Lang [2]).

Vector space	Kernel	Gaussian
Independence	Image	Triangular
Linear combination	Inverse	Gram–Schmidt
Span	Transpose	Eigenvector
Basis	Orthogonal	Singular value
Subspace	Operator	Decomposition
Projection	Diagonalization	LU form
Matrix	Normal form	Norm
Dimension	Eignvalue	Condition
Linear transformation	Similarity	Isomorphism
Rank	Diagonalize	Determinant

This is not something unique to linear algebra – a similar situation holds for many undergraduate mathematics courses. Certainly the authors of undergraduate texts do not share this student view of mathematics. In fact, the variety ways in which different authors organize their texts reflects the individual ways in which they have conceptualized introductory linear algebra courses. The wide variability that can be seen in a perusal of the many linear algebra texts that are used is a reflection the many ways that mathematicians think about linear algebra and their beliefs about how students can come to make sense of the content. Instruction in a course is based on considerations of content, pedagogy, resources (texts and other materials), and beliefs about teaching and learning of mathematics. The interplay of these ideas shaped our research project.

We deliberately mention two authors with clearly differing perspectives on an undergraduate linear algebra course: Strang's organization of the material takes an applied or application perspective, while Lang views the material from more of a "pure mathematics" perspective. A review of the wide variety of textbooks to classify and categorize the different views of the subject would reveal a broad variety of perspectives on the teaching of the subject. We have taken a view that seeks to go beyond the mathematical content to integrate current theoretical perspectives on the teaching and learning of undergraduate mathematics. Our project used integration of mathematical content, applications, and learning theories to provide enhanced learning experiences using rich content, student meta cognition, and their own experience and intuition. The project also used co-teaching and collaboration among faculty with expertise in a variety of areas including mathematics, computer science and mathematics education.

If one moves beyond the organization of the content of textbooks we find that at their heart they do cover a common core of the key ideas of linear algebra–all including fundamental concepts such as vector space and linear transformation. These observations lead to our key question "How is one to think about this task of organizing instruction to optimize learning?"

In our work we focus on the conception of linear algebra that is developed by the student and its relationship with what we reveal about our own understanding of the subject. It seems that even in cases where researchers consciously study the teaching and learning of linear algebra (or other mathematics topics) the questions are "What does it mean to understand linear algebra?" and "How do I organize instruction so that students develop that conception as fully as possible?" In broadest terms, our work involves (a) simultaneous study of linear algebra and learning theories, (b) having students connect learning theories to their study of linear algebra, and (c) the use of parallel mathematics and education courses and integrated workshops.

As students simultaneously study mathematics and learning theory related to the study of mathematics, we expect that reflection or meta cognition on their own learning will enable them to construct deeper and more meaningful understanding in both domains. We chose linear algebra for several reasons: It has not been the focus of as much instructional research as calculus, it involves abstraction and proof, and it is taken by many students in different programs for a variety of reasons. It seems to us to involve important mathematical content along with rich applications, with abstraction that builds on experience and intuition.

In our pilot study we taught parallel courses: The regular upper division undergraduate linear algebra course and a seminar in learning theories in mathematics education. Early in the project we also organized an intensive three-day workshop for teachers and prospective teachers that included topics in linear algebra and examination of learning theory. In each case (two sets of parallel courses and the workshop) we had students reflect on their learning of linear algebra content and asked them to use their own learning experiences to reflect on the ideas about teaching and learning of mathematics.

Students read articles – in the case of the workshop, this reading was in advance of the long weekend session – drawn from mathematics education sources including [3–10].

APOS (Action, Process, Object, Schema) is a theoretical framework that has been used by many researchers who study the learning of undergraduate and graduate mathematics [10,11]. We include a sketch of the structure of this framework and refer the reader to the literature for more detailed descriptions. More detailed and specific illustrations of its use are widely available [12]. The APOS Theoretical Framework involves four levels of understanding that can be described for a wide variety of mathematical concepts such as *function*, *vector space*, *linear transformation*: Action, Process, Object (either an encapsulated process or a thematicized schema), Schema (Intra, inter, trans – triad stages of schema formation). Genetic decomposition is the analysis of a particular concept in which *developing understanding* is described as a dynamic process of mental constructions that continually develop, abstract, and enrich the structural organization of an individual's knowledge.

We believe that students' simultaneous study of linear algebra along with theoretical examination of teaching and learning – particularly on what it means to develop conceptual understanding in a domain – will promote learning and understanding in both domains. Fundamentally, this reflects our view that conceptual understanding in any domain involves rich mental connections that link important ideas or facts, increasing the individual's ability to relate new situations and problems to that existing cognitive framework. This view of conceptual understanding of mathematics has been described by various prominent math education researchers such as Hiebert and Carpenter [6] and Hiebert and Lefevre [7].

2. Action–Process–Object–Schema theory (APOS)

APOS theory is a theoretical perspective of learning based on an interpretation of Piaget's constructivism and poses descriptions of mental constructions that may occur in understanding a mathematical concept. These constructions are called Actions, Processes, Objects, and Schema.

An **action** is a transformation of a mathematical object according to an explicit algorithm seen as externally driven. It may be a manipulation of objects or acting upon a memorized fact.

When one reflects upon an action, constructing an internal operation for a transformation, the action begins to be interiorized. A **process** is this internal transformation of an object. Each step may be described or reflected upon without actually performing it. Processes may be transformed through reversal or coordination with other processes.

There are two ways in which an individual may construct an **object**. A person may reflect on actions applied to a particular process and become aware of the process as a totality. One realizes that transformations (whether actions or processes) can act on the process, and is able to actually construct such transformations. At this point, the individual has reconstructed a process as a cognitive object. In this case we say that the process has been *encapsulated* into an object. One may also construct a cognitive object by reflecting on a *schema*, becoming aware of it as a totality. Thus, he or she is able to perform actions on it and we say the individual has *thematized* the schema into an object. With an object conception one is able to de-encapsulate that object back into the process from which it came, or, in the case of a thematized schema, unpack it into its various components. Piaget and Garcia [13] indicate that thematization has occurred when there is a change from usage or implicit application to consequent use and conceptualization.

A **schema** is a collection of actions, processes, objects, and other previously constructed schemata which are coordinated and synthesized to form mathematical structures utilized in problem situations. Objects may be transformed by higher-level actions, leading to new processes, objects, and schemata. Hence, reconstruction continues in evolving schemata.

To illustrate different conceptions of the APOS theory, imagine the following 'teaching' scenario. We give students multi-part activities in a technology supported environment. In particular, we assume students are using Maple in the computer lab. The multi-part activities, focusing on vectors and operations, in Maple begin with a given Maple code and drawing. In case of scalar multiplication of the vector, students are asked to substitute one parameter in the Maple code, execute the code and observe what has happened. They are asked to repeat this activity with a different value of the parameter. Then students are asked to predict what will happen in a more general case and to explain their reasoning. Similarly, students may explore addition and subtraction of vectors. In the next part of activity students might be asked to investigate about the commutative property of vector addition.

Based on APOS theory, in the first part of the activity – in which students are asked to perform certain operation and make observations – our intention is to induce each student's *action conception* of that concept. By asking students to imagine what will happen if they make a certain change – but do not physically perform that change – we are hoping to induce a somewhat higher level of students' thinking, the *process level*. In order to predict what will happen students would have to imagine performing the action based on the actions they performed before (reflective abstraction). Activities designed to explore on vector addition properties require students to encapsulate the process of addition of two vectors into an object on which some other action could be performed. For example, in order for a student to conclude that $u + v = v + u$, he/she must encapsulate a process of adding two vectors $u + v$ into an object (resulting vector) which can further be compared [action] with another vector representing the addition of $v + u$.

As with all theories of learning, APOS has a limitation that researchers may only observe externally what one produces and discusses. While schemata are viewed as dynamic, the task is to attempt to take a snap shot of understanding at a point in time using a genetic decomposition. A genetic decomposition is a description by the researchers of specific mental constructions one may make in understanding a mathematical concept. As with most theories (economics, physics) that have restrictions, it can still be very useful in describing what is observed.

3. Initial research

In our preliminary study we investigated three research questions:

- *Do participants make connections between linear algebra content and learning theories?*
- *Do participants reflect upon their own learning in terms of studied learning theories?*

- *Do participants connect their study of linear algebra and learning theories to the mathematics content or pedagogy for their mathematics teaching?*

In addition to linear algebra course activities designed to engage students in explorations of concepts and discussions about learning theories and connections between the two domains, we had students construct concept maps and describe how they viewed the connections between the two subjects. We found that some participants saw significant connections and were able to apply APOS theory appropriately to their learning of linear algebra.

For example, here is a sketch outline of how one participant described the elements of the APOS framework late in the semester. The student showed a reasonable understanding of the theoretical framework and then was able to provide an example from linear algebra to illustrate the model. The student's description of the elements of APOS:

Action: "Students' approach is to apply 'external' rules to find solutions. The rules are said to be external because students do not have an internalized understanding of the concept or the procedure to find a solution."

Process: "At the process level, students are able to solve problems using an internalized understanding of the algorithm. They do not need to write out an equation or draw a graph of a function, for example. They can look at a problem and understand what is going on and what the solution might look like."

Object level as performing actions on a process: "At the object level, students have an integrated understanding of the processes used to solve problems relating to a particular concept. They understand how a process can be transformed by different actions. They understand how different processes, with regard to a particular mathematical concept, are related. If a problem does not conform to their particular action-level understanding, they can modify the procedures necessary to find a solution."

Schema as a 'set' of knowledge that may be modified: "Schema – At the schema level, students possess a set of knowledge related to a particular concept. They are able to modify this set of knowledge as they gain more experience working with the concept and solving different kinds of problems. They see how the concept is related to other concepts and how processes within the concept relate to each other."

She used the ideas of determinant and basis to illustrate her understanding of the framework. (Another student also described how student recognition of the recursive relationship of computations of determinants of different orders corresponded to differing levels of understanding in the APOS framework.)

Action conception of determinant: "A student at the action level can use an algorithm to calculate the determinant of a matrix. At this level (at least for me), the formula was complicated enough that I would always check that the determinant was correct by finding the inverse and multiplying by the original matrix to check the solution."

Process conception of determinant: "The student knows different methods to use to calculate a determinant and can, in some cases, look at a matrix and determine its value without calculations."

Object conception: "At the object level, students see the determinant as a tool for understanding and describing matrices. They understand the implications of the value of the determinant of a matrix as a way to describe a matrix. They can use the determinant of a matrix (equal to or not equal to zero) to describe properties of the elements of a matrix."

Triad development of a schema (intra, inter, trans): "A singular concept – basis. There is a basis for a space. The student can describe a basis without calculation. The student can find different types of bases (column space, row space, null space, eigenspace) and use these values to describe matrices."

The descriptions of components of APOS along with examples illustrate that this student was able to make valid connections between the theoretical framework and the content of linear algebra. While the descriptions may not match those that would be given by scholars using APOS as a research framework, the student does demonstrate a recognition of and ability to provide examples of how understanding of linear algebra can be organized conceptually as more that a collection of facts.

As would be expected, not all participants showed gains in either domain. We viewed the results of this study as a proof of concept, since there were some participants who clearly gained from the experience. We also recognized that there were problems associated with the implementation of our plan. To summarize our findings in relation to the research questions:

- *Do participants make connections between linear algebra content and learning theories?* Yes, to widely varying degrees and levels of sophistication.
- *Do participants reflect upon their own learning in terms of studied learning theories?* Yes, to the extent possible from their conception of the learning theories and understanding of linear algebra.
- *Do participants connect their study of linear algebra and learning theories to the mathematics content or pedagogy for their mathematics teaching?*

Participants describe how their experiences will shape their own teaching, but we did not visit their classes.

Of the 11 students at one site who took the parallel courses, we identified three in our case studies (a detailed report of that study is presently under review) who demonstrated a significant ability to connect learning theories with their own learning of linear algebra. At another site, three teachers pursuing math education graduate studies were able to varying degrees to make these connections – two demonstrated strong ability to relate content to APOS and described important ways that the experience had affected their own thoughts about teaching mathematics.

Participants in the workshop produced richer concept maps of linear algebra topics by the end of the weekend. Still, there were participants who showed little ability to connect material from linear algebra and APOS. A common misunderstanding of the APOS framework was that increasing levels corresponded to increasing difficulty or complexity. For example, a student might suggest that computing the determinant of a 2×2 matrix was at the action level, while computation of a determinant in the 4×4 case was at the object level because of the increased complexity of the computations. (Contrast this with the previously mentioned student who observed that the object conception was necessary to recognize that higher dimension determinants are computed recursively from lower dimension determinants.)

We faced more significant problems than the extent to which students developed an understanding of the ideas that were presented. We found it very difficult to get students – especially undergraduates – to agree to take an additional course while studying linear algebra. Most of the participants in our pilot projects were either mathematics teachers or prospective mathematics teachers. Other students simply do not have the time in their schedules to pursue an elective seminar not directly related to their own area of interest. This problem led us to a new project in which we plan to integrate the material on learning theory – perhaps implicitly for the students – in the linear algebra course. Our focus will be on working with faculty teaching the course to ensure that they understand the theory and are able to help ensure that course activities reflect these ideas about learning.

4. Continuing research

Our current *Linear Algebra in New Environments* (LINE) project focuses on having faculty work collaboratively to develop a series of modules that use applications to help students develop conceptual understanding of key linear algebra concepts. The project has three organizing concepts:

- Promote enhanced learning of linear algebra through integrated study of mathematical content, applications, and the learning process.
- Increase faculty understanding and application of mathematical learning theories in teaching linear algebra.
- Promote and support improved instruction through co-teaching and collaboration among faculty with expertise in a variety of areas, such as education and STEM disciplines.

For example, computer and video graphics involve linear transformations. Students will complete a series of activities that use manipulation of graphical images to illustrate and help them move from action and process conceptions of linear transformations to object conceptions and the development of a linear transformation schema. Some of these ideas were inspired by material in Judith Cederberg's geometry text [14] and some software developed by David Meel, both using matrix representations of geometric linear transformations. The modules will have these characteristics:

- Embed learning theory in linear algebra course for both the instructor and the students.
- Use applied modules to illustrate the organization of linear algebra concepts.
- Applications draw on student intuitions to aid their mental constructions and organization of knowledge.
- Consciously include meta-cognition in the course.

To illustrate, we sketch the outline of a possible series of activities in a module on geometric linear transformations. The faculty team – including individuals with expertise in mathematics, education, and computer science – will develop a series of modules to engage students in activities that include reflection and meta cognition about their learning of linear algebra. (The Appendix contains a more detailed description of a module that includes these activities.)

Task 1: Use Photoshop or GIMP to manipulate images (rotate, scale, flip, shear tools). Describe and reflect on processes. This activity uses an ACTION conception of transformation.

Task 2: Devise rules to map one vector to another. Describe and reflect on process. This activity involves both ACTION and PROCESS conceptions.

Task 3: Use a matrix representation to map vectors. This requires both PROCESS and OBJECT conceptions.

Task 4: Compare transform of sum with sum of transforms for matrices in Task 3 as compared to other non-linear functions. This involves ACTION, PROCESS, and OBJECT conceptions.

Task 5: Compare pre-image and transformed image of rectangles in the plane – identify software tool that was used (from Task 1) and how it might be represented in matrix form. This requires OBJECT and SCHEMA conceptions.

Education, mathematics and computer science faculty participating in this project will work prior to the semester to gain familiarity with the APOS framework and to identify and sketch potential modules for the linear algebra course. During the semester, collaborative teams of faculty continue to develop and refine modules that reflect important concepts, interesting applications, and learning theory: Modules will present activities that help students develop important concepts rather than simply presenting important concepts for students to absorb.

The researchers will study the impact of project activities on student learning: We expect that students will be able to describe their knowledge of linear algebra in a more conceptual (structured) way during and after the course. We also will study the impact of the project on faculty thinking about teaching and learning: As a result of this work, we expect that faculty will be able to describe both the important concepts of linear algebra and how those concepts are mentally developed and organized by students. Finally, we will study the impact on instructional practice: Participating faculty should continue to use instructional practices that focus both on important content and how students develop their understanding of that content.

5. Summary

Our preliminary study demonstrated that prospective and practicing mathematics teachers were able to make connections between their concurrent study of linear algebra and of learning theories relating to mathematics education, specifically the APOS theoretical framework. In cases where the participants developed understanding in both domains, it was apparent that this connected learning strengthened understanding in both areas. Unfortunately, we were unable to encourage undergraduate students to consider studying both linear algebra and learning theory in separate, parallel courses. Consequently, we developed a new strategy that embeds the learning theory in the linear algebra course through the use of carefully designed modules that combine important linear algebra concepts, related student intuitions and experiences, and a view of the development of mathematical concepts described by the APOS framework. Mathematics instructors work with mathematics educators and computer scientists to consciously develop, implement and refine materials incorporating all three dimensions. While the faculty will explicitly identify the connections, students in the linear algebra course mostly encounter the learning theory implicitly as they reflect on their own learning. We expect that these materials will promote deeper conceptual understanding of key linear algebra concepts by students and enhanced insights into the connections between instruction and learning by the course instructors and developers.

Appendix: Sample module

A sample module is provided to illustrate the use of APOS to describe how an individual comes to understand the concept of *linear transformation* in increasingly sophisticated and generalized ways. This module could be used both with students in a linear algebra class, or with faculty to help them reflect on the APOS theoretical framework. While the module may require several hours of class time for students, in the context of faculty development the focus will be more on the integration of the learning theory and linear algebra, not on content, and so will require much less time.

Ma [15] interviewed American and Chinese teachers, asking them to describe arithmetic algorithms such as subtraction and multiplication. She found that a troubling number of American teachers could give the algorithm but could not explain why it works. In such cases, the teachers often were also unable to recognize that alternate algorithms – such as might be given by a student – also produced valid results. Her findings, based on teacher and student understanding of elementary arithmetic, have a theme that parallels the intent of our project. We will use problems involving core concepts of linear algebra to help faculty and students examine and reflect upon both the content (procedures and concepts) of linear algebra and how those ideas are developed to increasingly sophisticated and generalized levels by an individual.

The initial activity in this module (at the *action* level) asks the individual to transform an image using a software tool, reflecting on which characteristics of the image are preserved and which are lost through the mapping. Next, participants carry out mathematical actions producing geometric transformations, developing a *process conception* of transformation. Finally, the learner explores how matrix multiplication can be used to encapsulate the *process* of a given linear transformation, and also begins to reflect on distinctions between linear and non-linear transformations. These ideas require the development of an *object-level conception* of linear transformation. Reflection on modules such as this, along with readings selected from the research literature, will be used to help faculty participants achieve the goals of the LINE project in the teaching of their linear algebra courses. Similar faculty discussions will be used throughout the course to help ensure that students come to recognize how their understandings of core concepts develop during the semester.

In the sample module given below, the boxed text represents the student directions; annotations provide a discussion of how learning theory influences the design of this module.

Learning goals of the module

This module is designed to help students understand the meaning and properties of linear transformations, specifically:

- Matrix multiplication;
- What is 'transformation?' What is 'linear?'; and
- Connections between algebra and geometry

Pre-task for faculty: In advance, faculty read *How people learn: Brain, mind, experience, and school* [16] and other readings on mathematics learning and teaching and several APOS research papers, and discuss how these learning theories can relate to linear algebra and the study of mathematics. They collaboratively apply these theories to create a genetic decomposition for linear transformation, and reflect on and discuss the role of learning theories in designing the decomposition. They then collaboratively develop a module that builds and reinforces understandings of the linear transformation object conception (as described in APOS). Their results may resemble the module we present below.

Task 1 – Objective: To develop a geometric intuition for linear transformations.

Working individually, use image manipulation software such as *Photoshop* or *GIMP* to load a picture of a face. Experiment with the rotate, scale, flip, and shear tools.

Write a description of what each tool does. Explain the controls for each tool. (e.g., what is "Center X" in the rotation tool?)

Try to use these tools to change part of the image while leaving another part unchanged. What characteristics are common to all three tools?

How do you think these images are stored and displayed on the computer? How do you think the tools are programmed?

With your group, exchange your descriptions and explanations. Choose a description for sharing with the rest of the class, or combine elements.

Return to the class and report from your group the main points you have agreed upon.

The first four activities above correspond to an *action* level of the APOS framework, although the representation of transformations is an alternative one, with the software tool deployed to increase student engagement; activities five and six require thinking at a *process* level, in which students are asked to reflect on commonalities and relationships among the transformations.

Task 2 – Objective: To develop the idea of transformations of vectors.

Give a rule to map the vector $(1, 2)$ to $(-2, 4)$. What is the result when you apply the rule to the vector $(2, 5)$? Compare your rule with your neighbor's: Did you develop the same rule? How do you know it's the same or different?

Find a rule that maps $(1, 2)$ to $(-2, 4)$ and also maps $(0, 1)$ to $(-3, 3)$. What is the result when the rule is applied to $(3, 3)$?

Find a rule that maps $(1, 2)$ to $(-2, 4)$ and also maps $(0, 1)$ to $(0, 2)$. What is the result when the rule is applied to $(3, 3)$?

Apply each of the two rules above to the triangle with corners at $(1, 2)$, $(0, 1)$, and $(1, 0)$. What does each rule do to the shape? Compare and contrast the two transformations.

Apply each of the two rules above to 4 points on the line $y = 2x$. What do you notice?

What would each of the two rules above do to a circle centered at the origin?

Find a rule that will map $(0, 1)$ to $(1, 2)$ and $(1, 0)$ to $(1, -1)$? Explain the reasoning that helped you to find this rule.

The activities in Task 2 above go beyond the action level: students are given a result and asked to find a way to achieve that result. They are asked to develop, describe, and reflect rather than simply to act. This promotes a *process* conception of linear transformation.

Task 3 – Objective: To develop the notion of matrix notation of vector transformations.

One way of describing vector mappings is to give a general expression for the mapping of the point (x, y). For example, we might say that each (x, y) is mapped to $(x + y, 2y - x)$. According to this rule, $(0, 1)$ is mapped to $(0 + 1, 2 * 1 - 0) = (1, 2)$. Using the same rule, find the images of $(1, 0)$ and $(3, 4)$.

Notice that this could be written in matrix form as well: We could express the mapping as $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$.

What is the result of $A\begin{bmatrix} x \\ y \end{bmatrix}$? How does this compare with our first description of the rule?

Using vectors, illustrate and describe what's happening geometrically.

Can we express the mapping from $(1, 2)$ to $(-2, 4)$ and $(0, 1)$ to $(0, 2)$ in this form? To do this, we need to find the matrix A with elements $a\ b\ c\ d$ so that $A(1, 2) = (-2, 4)$ and $A(0, 1) = (0, 2)$. We could write this more compactly as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 4 & 2 \end{bmatrix}$. What four equations in a, b, c, and d would help you find this matrix?

The activities of Task 3 relate matrix multiplication to linear transformation, starting to move students toward an *object* conception of linear transformation. Students need to be able to build up the process of transforming a vector using matrix multiplication, and then break down that process in order to perform activity four, above.

Task 4 – Objective: To examine properties of linear transformations.

Pick two 2-dimensional vectors. Compute and draw a diagram of their vector sum. Apply the transformation A (above in Task 3) to this sum.

Apply the transformation A to the two vectors you picked, compute and draw the sum of the transformed vectors. What do you notice about the relationship between the transformed sum and the sum of the transforms?

Does the function $f(x) = x^2$ have this property? Or $f(x) = sqrt(x)$? Why or why not? Which functions in one variable do have this property? How do you know? Explain your reasoning.

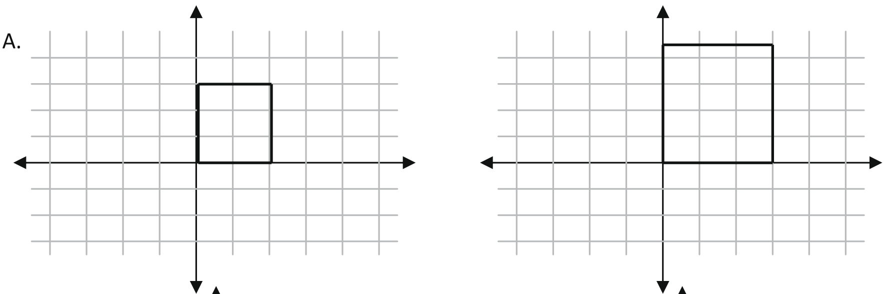
Why do you think these transformations are called linear transformations?

While the first four activities of Task 4 are at the action level, they are designed to promote the object conception of linear transformation that supports the reflection necessary to complete activities five and six of this task.

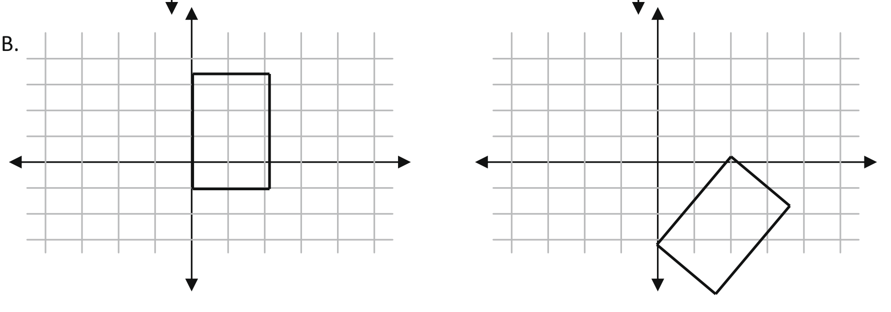
Task 5 – Objective: To reflect on matrix representation of common linear transformations.

Recall from Task 1 the four GIMP tools used (scale, shear, rotate, flip). For each pair of figures below, identify which tool was most likely used to perform the illustrated transformation, then try to give a matrix for the transformation.

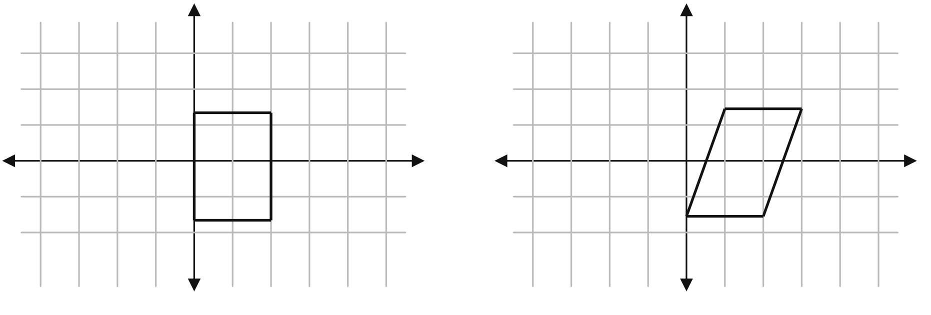
A.



B.



C.



Task 5 requires an object conception of linear transformation.

References

[1] Gilbert Strang, Linear Algebra and its Applications, fourth ed., Thomson/Brooks Cole, Belmont CA, 2006
[2] Serge Lang, Linear Algebra, third ed., Springer, New York, 2004.
[3] M.C. Wittrock (Ed.), Handbook of Research on Teaching, Macmillan, New York, 1986.
[4] National Research Council, How People Learn: Brain, Mind, Experience, and School, National Academy Press, Washington DC, 2000.
[5] J.D. Novak, D.B. Gowin, Learning How to Learn, Cambridge University Press, Cambridge, UK, 1984.
[6] J. Hiebert, T.P. Carpenter, Learning and teaching with understanding, in: D. Grouws (Ed.), Handbook of Research on Mathematics Teaching and Learning, MacMillan, New York, 1992, pp. 65–97.
[7] J. Hiebert, P. Lefevre, Conceptual and procedural knowledge in mathematics: an introductory analysis, in: J. Hiebert (Ed.), Conceptual and Procedural Knowledge: The Case of Mathematics, Erlbaum, Hillsdale, NJ, 1987.
[8] D.C. Grouws (Ed.), Handbook of Research on Mathematics Teaching and Learning, Macmillan, New York, 1992.
[9] B. Baker, L. Cooley, M. Trigueros, The schema triad – a calculus example, Journal for Research in Mathematics Education 31 (5) (2000) 557–578.
[10] M. Asiala, A. Brown, D. DeVries, E. Dubinsky, D. Mathews, K. Thomas, A framework for research and curriculum development in undergraduate mathematics education, in: J. Kaput, A.H. Schoenfeld, & E. Dubinsky (Eds.), CBMS Issues in Mathematics Education 6: Research in Collegiate Mathematics Education II, 1996, pp. 1–32.
[11] L. Cooley, Reflective abstraction and writing in calculus, J. Math. Behav. 21 (3) (2002) 255–282.
[12] L. Cooley, M. Trigueros, B. Baker, Schema thematization: a framework and an example, J. Res. Math. Ed. 38 (4) (2007) 370–392.
[13] J. Piaget, R. Garcia, Psychogenesis and the History of Science, Columbia University Press, New York, 1989, J. Feider, Trans., original work published 1983.
[14] J. Cederberg, A Course in Modern Geometries, second ed., Springer-Verlag, New York, 2005.
[15] L. Ma, Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Mathematics in China and the United States, Lawrence Erlbaum Associates, Mahway, NJ, 1999.
[16] J.D. Bransford, A.L. Brown, R.R. Cocking (Eds.), How People Learn: Brain, Mind, Experience, and School, National Academy Press, Washington, DC, 1999.